MATHEMATICAL MODEL OF A GAS CONDENSING INTO A SOLID STATE ON A FINNED SURFACE

A. K. Zhebrovskii and M. K. Trubetskov

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For the problem of a gas condensing into a solid phase on a finned surface a mathematical model is proposed which leads to a boundary-value problem for a nonlinear second-order equation. A comparison with experiment is made and the problem on an optimal rectangular cross section fin is considered.

The main advantage of the condensation method of air evacuation in a pressure range below 40 torr is the possibility of obtaining, in a vacuum chamber, air free of dust and hydrocarbons. On pump panels, condensation proceeds into a solid phase, the initial gas-to-panel heat-transfer coefficient is usually about 50 W/(m^2·deg), and it quickly decreases with time because of the increasing thermal resistance of a growing cryoprecipitate layer. To reduce pump overall dimensions, it is necessary to have a cryopanel with a developed and actively air-evacuating surface.

Gas condensation on a finned surface has been investigated in many works [1-11]. The fundamental principles of classical theory and results of heat transfer theory in thin fins, without considering a condensation process, are a matter of concern in [1, 3]. A gas (vapor) condensation process on a finned surface into a liquid phase was examined in [2, 4-6, 8-10]. In this case, the condensate is removed by gravity from a system (condensation by gravity) and the problem is stationary. Condensation into a solid phase is discussed in [7, 11], which are of a purely experimental character, while the mathematical model considered in [7] is based on the assumption that a fin is isothermal, the condensate layer thickness is constant everywhere and therefore is rather rough.

1. We consider a single isolated fin with height L and half-thickness $\sigma(x)$. Considering the fin to be thin ($\sigma(x) \approx \sigma_0 << L$) and infinitely extended along the z-axis (a two-dimensional problem), we take as a model of a heat transfer process inside a fin covered with a condensate layer, a one-dimensional (with respect to x) differential equation for a fin temperature averaged along y, which is obtained from the following considerations.

Since the thermal conductivity of the fin is much higher than the condensate counterpart, it is assumed that heat fluxes inside the fin mainly propagate in a longitudinal direction, while those inside the condensate move in a transverse direction from the condensate surface toward the fin (Fig. 1). We consider a layer between the planes x and x + dx and write the law of conservation of heat. Let T(x, t) be the mean fin temperature in the cross section x. Assume that the heat flux inside the fin is directed along the x-axis, the condensate layer is sufficiently thin and the heat flux along it may be neglected. Thus, the specific heat flux j_c carried by a condensing gas arrives, without losses, through a side surface of the fin. The heat balance equation for the indicated volume allows, as a consequence, a partial differential equation to be obtained

$$\lambda_f \frac{\partial}{\partial x}\left(\sigma \frac{\partial T}{\partial x}\right) + j_c = \sigma c_f \rho_f \frac{\partial T}{\partial t} \tag{1}$$

for the mean temperature.

We find an equation for the specific heat flux j_c by using the gas motion and condensation model. Neglecting natural convection, we consider only gas motion caused by its condensation. Considering the curvature radius of a condensate surface to be much larger than the diffusional layer thickness, we introduce a local coordinate system at each point of the surface (Fig. 2). Within the scope of the assumptions made, the gas motion is naturally considered to be locally one-dimensional and occurring

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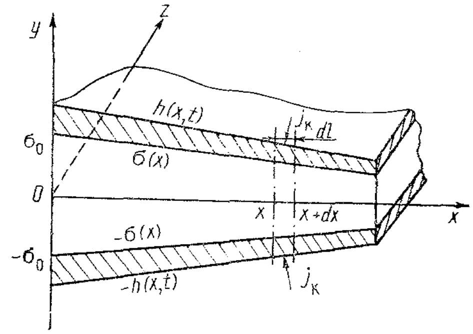

Fig. 1. The single fin with a condensate layer.

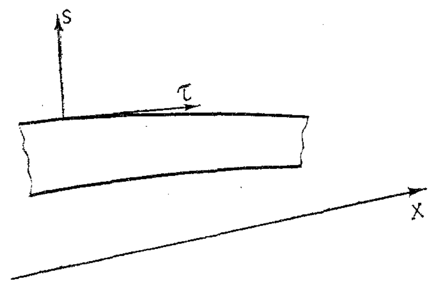

Fig. 2. Local coordinate system.

along the normal toward the condensate surface (the s-coordinate). From the continuity equation and the above assumptions it follows that the gas velocity v remains constant along the direction s and the heat-transfer process inside the gas is described by the equation

$$vT_s = \chi^2 T_{ss}, \ \chi = \frac{\lambda_g}{c_g \rho_g}, \ 0 < s < \infty, \tag{2}$$

with the boundary conditions

$$T(0) = T_{in}, \ T(+\infty) = T_\infty. \tag{3}$$

Equation (2) is written on the assumption that a heat transfer process slowly develops and the operator $\partial/\partial t$ may be neglected. If $v < 0$ (the flow is directed towards the condensate), then a solution of Eq. (2) with the boundary conditions (3) has the form typical of boundary layers:

$$T(s) = T_\infty + (T_{in} - T_\infty) \exp(vs/\chi). \tag{4}$$

Note that the expression (4) represents the dependence on time and on the y-coordinate in implicit form, since the local gas velocity v is a function of the above variables.

The laws of conservation on the condensate surface result in the following relations. The law of conservation of matter leads to an obvious equation (the function $y = h(x, t)$ describes the condensate surface; see Fig. 1)

$$\frac{\partial h}{\partial t} = -\frac{\rho_g}{\rho_c} v, \tag{5}$$

the law of conservation of heat needs a more detailed consideration.

Consider a section of the condensate surface with the length dl (Fig. 1). The amount of heat carried by the gas is equal to

$$Q_g = -c_g \rho_g T|_{s=0} v dt dl + \lambda_g \frac{\partial T}{\partial s}\Big|_{s=0} dt dl,$$

The heat released on account of a phase transition is

$$Q_p = r\rho_c dh dl.$$

The heat flux toward the fin from the condensate surface is j_c, since according to the assumptions made above the longitudinal heat fluxes may be neglected. This means that the amount of heat carried away is

$$Q_c = j_c dx dt.$$

Inside a volume of substance caused by condensation, the amount of remaining heat is

$$Q_h = c_c \rho_c T_{in} dh dl.$$

From the balance relation

$$Q_c + Q_h = Q_p + Q_g$$

after simple transformations, taking into account (4) and (5), we arrive at the expression for the flux j_c:

$$j_c = \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2} \rho_c \frac{\partial h}{\partial t} \tilde{r}, \quad \tilde{r} = r + c_g T_\infty - c_c T_{in}. \tag{6}$$

One more equation may be obtained from the next considerations. According to the assumptions made, the flux j_c does not depend on y in the condensate layer at each fixed x or, otherwise, the temperature in the condensate layer is a linear function of depth. It yields another expression for the flux j_c:

$$j_c = \lambda_c \frac{T_{in} - T}{h - \sigma}. \tag{7}$$

After elimination of the flux j_c from Eqs. (1), (6), and (7) we obtain a system of two partial differential equations for two unknown functions T(x, t) and h(x, t). Before formulating the boundary and initial conditions for this system, we introduce dimensionless quantities.

First, we introduce the dimensionless temperature Θ as follows

$$T = T_{in} + \Theta (T_b - T_{in}),$$

and instead of the function h(x, t) consider the condensate layer thickness $\delta(x, t) = h(x, t) - \sigma(x)$ having a more evident physical meaning. The dimensionless condensate layer thickness Δ is introduced as $\delta = L^2\lambda_c/\sigma_0\lambda_f$, and the independent variables X and τ as

$$X = \frac{x}{L}, \quad \tau = \frac{\lambda_f^2 (T_{in} - T_b)}{\rho_c L^4 \lambda_c \tilde{r}} t.$$

For a condensate layer not too thick, it may be assumed $\partial h/\partial x \approx d\delta/dx$; then the mathematical model, after changing the variables, may be reformulated in the form of the following system (a prime indicates the derivative with respect to the dimensionless coordinate X):

$$\frac{1}{\sigma}\frac{\partial}{\partial X}\left(\sigma \frac{\partial \Theta}{\partial X}\right) - \sqrt{1 + (\sigma'/L)^2}\,\frac{\sigma_0}{\sigma}\frac{\partial \Delta}{\partial \tau} = \frac{c_f \rho_f \lambda_f (T_{in} - T_b)\sigma_0^2}{L^2 \rho_c \lambda_c \tilde{r}} \frac{\partial \Theta}{\partial \tau}, \tag{8}$$

$$\sqrt{1 + (\sigma'/L)^2}\,\frac{\partial \Delta}{\partial \tau} = \frac{\Theta}{\Delta}. \tag{9}$$

In the case of thin fins for which $\sigma_0 << L (\rho_c \lambda_c \tilde{r}/[c_f \rho_f \lambda_f \times (T_{in} - T_b)])^{1/2}$, we may neglect the thermal lag of the fin and assume the right-hand side of Eq. (8) to be zero. Finally, we obtain

$$\frac{1}{\sigma}\frac{\partial}{\partial X}\left(\sigma \frac{\partial \Theta}{\partial X}\right) - \sqrt{1 + (\sigma'/L)^2}\,\frac{\sigma_0}{\sigma}\frac{\partial \Delta}{\partial \tau} = 0, \tag{10}$$

$$\sqrt{1 + (\sigma'/L)^2}\,\frac{\partial \Delta}{\partial \tau} = \frac{\Theta}{\Delta}. \tag{11}$$

It is necessary to supplement Eqs. (10), (11) with initial and boundary conditions. Assume that at the initial moment of time $\tau = 0$ a condensate is absent on the fin:

$$\Delta|_{\tau=0} = 0, \tag{12}$$

the fin base temperature is given and equal to T_b and the fin apex is heat-insulated. The last condition is the standard one in the theory of heat transfer of finned surfaces, since heat transfer near the fin apex insignificantly contributes to the heat transfer process on the whole. Thus

$$\Theta|_{X=0} = 1, \quad \Theta'|_{X=1} = 0. \tag{13}$$

Equations (10) and (11) with the initial and boundary conditions (12) and (13) form a mathematical model of gas condensation on an isolated fin.

2. Owing to the fact that the partial derivatives of functions Θ and Δ with respect to different independent variables enter into Eqs. (10), (11), these equations are reduced to one nonlinear differential equation. We rewrite the Eq. (11) in the form

$$\sqrt{1+(\sigma'/L)^2}\,\frac{1}{2}\frac{\partial \Delta^2}{\partial \tau} = \Theta \tag{14}$$

and integrate it and Eqs. (10) over τ using the initial condition (12). After introducing the new auxiliary function $\Psi(X, \tau) = \int_0^\tau \Theta(X, \tau')\,d\tau'$, Eq. (12) allows elimination of the function Δ from the system and finally yields the boundary-value problem for $\Psi(X, \tau)$:

$$\frac{\partial}{\partial X}\left(\sigma \frac{\partial \Psi}{\partial X}\right) = \left(1 + \frac{\sigma'^2}{L^2}\right)^{1/4} \sigma_0 \sqrt{2\Psi}, \tag{15}$$

$$\Psi|_{X=0} = \tau, \quad \left.\frac{\partial \Psi}{\partial X}\right|_{X=1} = 0. \tag{16}$$

The problem for the determination of the dimensionless temperature $\Theta(X, \tau) = \partial\Psi/\partial\tau$ is obtained from the boundary-value problem (15), (16) by varying it with respect to τ and is of the form

$$\frac{\partial}{\partial X}\left(\sigma \frac{\partial \Theta}{\partial X}\right) = \left(1 + \frac{\sigma'^2}{L^2}\right)^{1/4} \sigma_0 \frac{\Theta}{\sqrt{2\Psi}}, \tag{17}$$

$$\Theta|_{X=0} = 1, \quad \left.\frac{\partial \Theta}{\partial X}\right|_{X=1} = 0. \tag{18}$$

This linear boundary-value problem relative to Θ must be solved after determining the function Ψ from the problem (15), (16).

Problems (15)-(18) have been solved by the shooting method [12], while the auxiliary Cauchy problems, by the Runge–Kutta method of the fourth-order of accuracy.

3. We consider the case most often met with in practice, i.e., rectangular cross section fins. For such fins, $\sigma'(X) \equiv 0$, and Eqs. (15), (16) are simplified as

$$\frac{\partial^2 \Psi}{\partial X^2} = \sqrt{2\Psi}, \quad \Psi|_{X=0} = \tau, \quad \left.\frac{\partial \Psi}{\partial X}\right|_{X=1} = 0. \tag{19}$$

We obtain an approximate solution of the problem (19) by linearizing the differential equation in the vicinity of $\Psi \simeq \tau$. To correct $\Omega = \Psi - \tau$, we obtain the linear boundary-value problem

$$\frac{\partial^2 \Omega}{\partial X^2} - \frac{\Omega}{\sqrt{2\Psi}} = \sqrt{2\tau}, \quad \Omega|_{X=0} = 0, \quad \left.\frac{\partial \Omega}{\partial X}\right|_{X=1} = 0,$$

whose solution is found in explicit form. Finally, we obtain for the function Ψ

$$\Psi = 2\tau \frac{\operatorname{ch} \varkappa (1 - X)}{\operatorname{ch} \varkappa} - \tau. \tag{20}$$

Comparison with numerical calculations shows that the approximation (20) has a relative error not worse than 3% if the dimensionless time $\tau \geq 0.8$; here the error monotonically decreases with increasing τ. Thus, the approximate solution obtained may be used at least for qualitative analysis of the characteristics of rectangular fins. Comparison of a numerical solution of the problem for rectangular fins and experimental data is a matter of concern in the next section.

As an example of using the mathematical model obtained, we consider a problem on the choice of optimal parameters of a rectangular fin. The problem is aimed at obtaining a maximum possible condensate mass on the fin at a given moment of time t. The running condensate mass on the fin is

$$m = 2\rho_c \int_0^L \delta(x, t)\, dx,$$

or in terms of new variables

$$m = C \int_0^1 \Delta(X, \tau)\, dX, \quad C = \frac{2\rho_c L^3 \lambda_c}{\sigma_0 \lambda_f}. \tag{21}$$

Taking (14) and (19) into account, we have for a rectangular fin $\Delta(X, \tau) = \sqrt{2\Psi} = \partial^2\Psi/\partial X^2$ which, after substituting into (21) yields a more convenient expression

$$m = -C \left.\frac{\partial \Psi}{\partial X}\right|_{X=0}. \tag{22}$$

In the case of using the approximate solution obtained above by the linearization method, the criterion (22) may be transformed to the form

$$m = \frac{2\rho_c L^3 \lambda_c}{\sigma_0 \lambda_f} (2\tau)^{3/4} \operatorname{th} (2\tau)^{-1/4}. \tag{23}$$

We analyze the dependence of the criterion (23) on the geometric parameters. For this, we substitute the expression for dimensionless time τ in terms of physical time t and let C_1 and C_2 denote the combinations of physical quantities independent of σ_0 and L. We obtain $m = C_1\sigma_0^{1/2}\tanh(C_2 L\sigma_0^{-1/2})$, whence it is seen that the condensate mass monotonically increases with L, but this growth soon slows down: with the growth of L the condensation process gets "saturated," i.e., the dependence m(L) has a flat asymptote. This is attributed to a rapid increase of the thermal resistance of the fin and, as a consequence, to a decrease of its effectiveness. The dependence on σ_0 is of a similar character. The presence of the flat asymptote is explained by the fact that the model does not include the processes at the end plane of the fin; thus an increase in σ_0 does not result in an increase of the working area of the fin though causes a decrease of thermal resistance. These conclusions are quite consistent with simple qualitative considerations concerning the condensation process: the larger the fin dimensions, the larger the running condensate mass on it to the given moment of time.

Obviously, the formulation of the problem of fin optimization must include a constant, for which we assume a cross-sectional area of the fin be

$$2\sigma_0 L = S = \text{const}. \tag{24}$$

The problem of function minimization (23) with the constraint (24) may be easily solved analytically. However, the subsequent analysis shows that its solution is found within the τ range at which the approximate solution (20) of the problem (19) has an error of more than 10%. Therefore, we consider a numerical algorithm of solution of the formulated problem.

Using (24), we eliminate, σ_0 from (22) and express m in terms of τ:

$$m = C_3 \tau^{-2/3} \left.\frac{\partial \Psi}{\partial X}\right|_{X=0}.$$

The maximum of the given function is determined by calculating its derivative and equating it to zero. In calculations, determination of the derivative of a boundary-value problem solution by the value of the boundary condition leads to the variation problem. The final result is

$$\frac{\partial^2 \Psi}{\partial X^2} = \sqrt{2\Psi}, \quad \Psi|_{X=0} = \tau, \quad \left.\frac{\partial \Psi}{\partial X}\right|_{X=1} = 0, \tag{25}$$

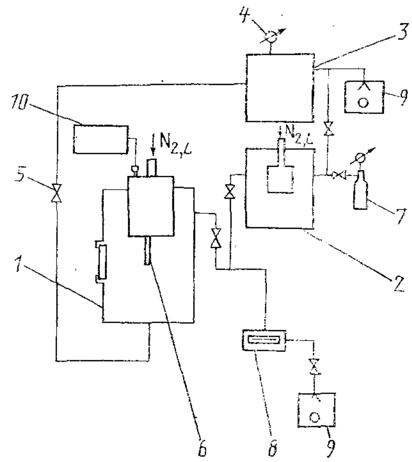

Fig. 3. Schematic of the experimental set-up: 1) vacuum chamber; 2) freezer; 3) receiver; 4) standard vacuum gauge; 5) leak; 6) fin; 7) gas bottle; 8) nitrogen trap; 9) vacuum pump; 10) vacuum gauges.

$$\frac{\partial^2 U}{\partial X^2} = \frac{U}{\sqrt{2\Psi}}, \quad U|_{X=0} = 1, \quad \left.\frac{\partial U}{\partial X}\right|_{X=1} = 0, \quad \left.\frac{\partial \Psi}{\partial X}\right|_{X=0} = -\frac{3}{2}\tau \left.\frac{\partial U}{\partial X}\right|_{X=0}. \tag{26}$$

We may show that the problem obtained allows us to determine a single value of the parameter τ at which the desired maximum is realized. The problem (25), (26) has been solved by the modified shooting method at which the auxiliary Cauchy problem for Eq. (25) is solved from right to left; next a current value of τ and initial conditions for U at the point X = 0 are determined and then the Cauchy problem for Eq. (26) is solved from left to right.

Numerical solution of the problem (25), (26) gives

$$\tau_{\text{opt}} = 0{,}3291 \text{ and } \left.\frac{\partial \Psi}{\partial X}\right|_{X=0 \text{ and } \tau=\tau_{\text{opt}}} = -0{,}5558.$$

Knowing τ_{opt} yields $L_{\text{opt}}^2\sigma_{0,\text{opt}} = \tau_{\text{opt}}^{-1/2}\lambda_f[(T_{\text{in}} - T_b)t/(\rho_c\lambda_c\bar{r})]$ which, together with (24), allows one to determine concrete optimal dimensions.

4. The basic calculated results have been verified on an experimental set-up (Fig. 3). A study has been made of time dependences of a cryoprecipitate profile on a fin, a fin temperature field, and local and integrated responsiveness of the fin. The experimental bench consisted of vacuum chamber 1 with a cryogenic surface, freezer 2, receiver 3, temperature, pressure and cryoprecipitate thickness measuring devices. Vacuum chamber 1 and receiver 3 were preliminarily evacuated by mechanical and adsorption pumps to a pressure not higher than $1 \cdot 10^{-5}$ torr. Prior to the experiment, a gas purified in the freezer came into the receiver by filling its volume to atmospheric pressure controlled by standard vacuum gauge 4. The vacuum chamber was manufactured as a cylinder with a visualization window. The window was intended for observation of the cryoprecipitate growth on the cryogenic surface and on some sections of a side surface of the fin. The flow rate of the gas from the receiver was regulated by leak 5 and measured against pressure drop by a standard manometer 4. To measure the gas pressure in the vacuum chamber, pressure gauges 13VT3-003 together with ampere voltmeters F-30 and VDO-1 were used. Temperature fields on the cryopanel and at different points of the fin 6 were measured by differential copper-constantan thermocouples together with Shch-300 voltmeters. The cryoprecipitate thickness on the fin was determined by thickness gauges (not indicated in the figure).

The set-up was brought into the operational mode in the following sequence. With gas supply to the chamber being valved off, the mechanical pump 2NVR-5DM and the sorption pump NKS-100 were connected to the latter in succession. These pumps allowed vacuum about $1 \cdot 10^{-5}$ torr to be obtained in the system. After cooling the cryopanel by liquid nitrogen, the pressure in the vacuum chamber was reduced to less than $1 \cdot 10^{-6}$ torr. Then the leak was employed to create the required gas pressure and the cryoprecipitate was frozen onto the fin and the cryopanel.

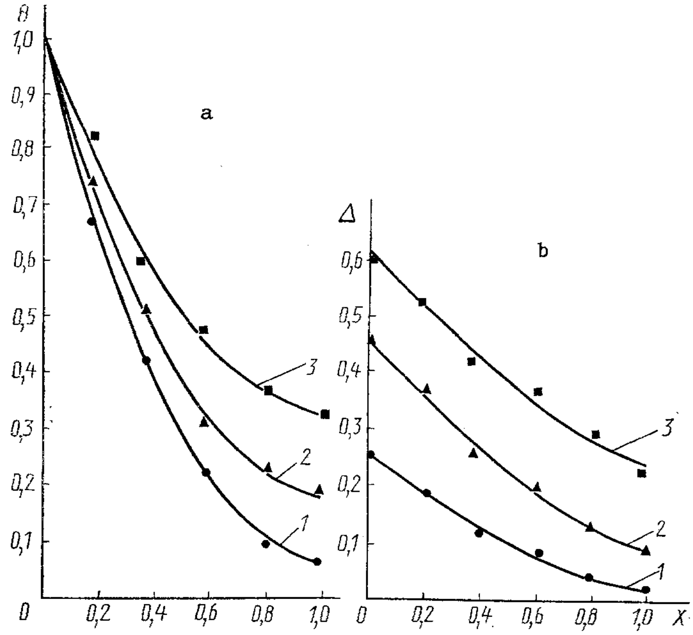

Fig. 4. Dimensionless temperature distribution in the fin (a) and condensate thickness profile (b) at different moments of time: 1) τ = 0.032; 2) 0.096; 3) 0.192.

We consider the results of one of the experiments in which an investigation was made of carbon dioxide condensation with chamber pressure 133 N/m^2 on a copper fin with height L =0.1 m and half-thickness $\sigma_0 = 0.65 \cdot 10^{-3}$ m cooled with liquid nitrogen (the fin base temperature was T_b = 80 K). The remaining parameters were as follows: T_{in} = 138 K, T_∞ = 290 K, c_g = 820 J/(kg·K), c_c = 900 J/(kg·K), r = $5.73 \cdot 10^5$ J/kg, ρ_c = 1670 kg/m^3, λ_c = 1.0 W/(m·K), λ_f = 500 W/(m·K). The process was studied within 1 h.

Figure 4 shows the profiles of dimensionless temperature distribution $\Theta(X)$ and dimensionless cryoprecipitate thickness distribution along the fin height $\Delta(X) = [2\Psi(X)]^{1/2}$ at different τ. Continuous curves are obtained using the proposed mathematical model, points indicate experimental values. A comparison of the given experimental and theoretical results reveals the adequacy of the proposed mathematical model and the considered physical process in a sufficiently wide range of input parameters. Whence it follows that the optimal solutions for rectangular fins may serve as a basis in designing effective cryogenic condensation systems.

NOTATION

L [m], fin height; σ [m], fin half-thickness; x, y, z [m], current coordinates; T [K], temperature; j [W/m^2], specific heat flux; λ [W/(m·K)], thermal conductivity; c [J(kg·K)], specific heat capacity; ρ [kg/m^3], density; t [sec], time; s [m], local coordinate; v [m/sec], gas velocity; χ [m^2/sec], thermal diffusivity; h [m], condensate thickness; l [m], length; r [J/kg], heat of desublimation; Θ, dimensionless temperature; Δ, cryoprecipitate dimensionless thickness; X, dimensionless length; τ, dimensionless time; m [kg/m], running mass; S [m^2], cross-sectional area. Indices: in, interphase; b, fin base; 0, initial value; f, fin; c, condensate; g, gas; ∞, at a distance from the interphase.

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